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ON THE FREQUENCY DISTRIBUTION OF THE ORBITAL ELEMENTS IN THE INTER-PLANETARY DUST PARTICLES

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ON THE FREQUENCY DISTRIBUTION OF THE ORBITAL ELEMENTS IN THE INTER-PLANETARY DUST PARTICLES.

by

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With 9 Figures within the text

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We have calculated the connection of the frequency distribution of the orbital elements (large semi-axis, excentricity, inclination) with the inter-planetary dust density and with the number of particles falling upon the earth, for a rotation-symmetrical dust cloud. We have derived for the frequency distribution a model that represents the density distribution of the dust on the basis of observations of the zodiacal light. The distribution of the orbital elements in the case of meteors, that is to be anticipated on the basis of the model, has been compared with the empirical material. When we consider the varying order of magnitude of the radius of zodiacal light particles (10^{-3}cm) and of meteors (10^{-1}cm), then the model will be satisfactory as a first approximation, particularly for the distribution of the inclination and for the distribution of the perihelion distances. When we use, in the case of massive incidence, the formula

$$M = Q \cdot U_{eff} \cdot d$$

When Q is the cross-section of the earth and d is the interplanetary density of the particles, then it will follow from the model that the value of the "effective velocity of the incidence", U_{eff}, will be in units of the velocity of the orbit of the earth - equal to 0.95.

Author

INTRODUCTION

Up to this time, little has been known as to the distribution function of the orbital elements of the dust-particles within the solar system. Statements relating to it may be obtained by three approaches: By way of the orbits of the sources of the dust; by way of the distribution of the orbital elements of meteors: and by way of the density distribution of the dust, which - in turn - may be found by means of observations of the zodiacal light. In this paper, the last approach will be considered in some detail, since the first two methods have not yet succeeded in providing sufficient information regarding the problem.

The comets (F.L. Whipple, 1955) as well as the small planets (S.Piotrowski 1954) are probably capable, due to their gradual decomposition, of replacing the dust, that disappears continously from inter-planetary space as a result of disturbances and of being gathered up by the sun and the planets. But the share supplied by each one of these groups is unknown. In addition, the frequency of the orbital elements in the sources is being observed in a distorted manner, due to the probability of their discovery. Finally, the orbits of the dust particles are subject-after they have been separated - to changes due to disturbances by planets, due to the Poynting-Robertson effect, and due to collisions with inter-planetary gases. For that reason, the results obtained by this method are very unreliable.

By using the collision probability of the particles with the planets (E.Oepik, 1951) it is possible to calculate the interplanetary frequency from the frequency of the orbital elements of the meteors. This was done, for the first time, for the orbital inclination of a small material of photographic orbits of meteors, by F.L. Whipple (1954). Within the immediate future, the number of exactly determined orbits of meteors will increase tremendously, on the basis of two-station observations by means of Super-Schmidt-meteor-cameras and of radar observations from three stations, so that it will be possible to use that method to a greater extent. But, it will supply data only for larger dust particles ($r. > 10^{-1}$ cm), and only for those that contribute to the density within the distance of the earth from the sun, But, there is the risk that, due to the collection of dust on the part of the earth, the density of the distribution may present a discontinuity at this location (E. Oepik, 1951). For that reason, we shall calculate - in the following investigation-the frequency of the orbital elements of the meteors by way of controlling our results, from an interplanetary distribution that has been obtained by the third one of the afore-mentioned methods.

Zodiacal light observations provide us with information on the dust density up to a solar distance of some 1.5A.U. Here, possible discontinuities of the density near the orbits of the planets play hardly any part. This independence of the special conditions near the earth is an essential advantage of the considerations being made here. It is true that assumptions regarding the scattering medium enter into the solution of the integral equation of the density on the basis of the luminosity of the zodiacal light. In earlier investigation (last by H. C. van de Hulst, 1947), the entire zodiacal light had been ascribed to scattering dust particles, so that a slow decrease of the density from the sun into an outward direction was the result obtained. On the other hand, A. Behr and H. Siedentopf (1953) as well as H. Elsässer (1954) explain the measured polarization of the zodiacal light on the basis of scattering on free electrons. The density share of the dust will then become smaller, particularly near the sun; it follows that the density within the ecliptic, from o.6 A.U. on, in an outward direction, is independent of the distance from the sun. This last model of the dust density has been used as the basis of the frequency distribution of the orbital elements as developed in the following.

By means of plausible assumptions, it is possible to limit the investigation to the distribution of the large semi-axes a, the eccentricities e, and the inclinations i. The relation between the density distribution in a rotation-symmetrical dust cloud as described by two variables, and of the distribution function of the three orbital elements will be established in the first part. Suggestions in regard to the method adopted were found in the afore-mentioned paper by E. Oepik (1915), whose collision formula will here be, once more the result, by means of a more general method on the basis of the density relation. The indefinite nature of the problem - determination of a function of three variables on the basis of a function of two variables - will become evident in the discussion of simple models in the second part, inasmuch as but few statements can be made as to the distribution of the eccentricities. But, it is hoped that this defect may be alleviated, to some extent, by the examination of the distribution function by means of the data relating to the meteors, as soon as more material in that respect has become known to us than we were able to use in the preliminary comparison of the sections of the second part.

I. Integrals for particle density and particle incidence on the earth.

1. Suppositions

In order to reduce the number of the parameters, we shall require suppositions in regard to the distribution function of the orbital elements of the interplanetary dust particles. For the sake of simplicity, we shall consider the distribution function to be independent of the particle size; it is true that this will be correct only as to the first approximation, due to the stronger effect of disturbances on small particles. The following equations will, however, apply also to each particle size separately.

For three of the six orbital elements, we made certain assumptions of even distribution: 1. For a certain form and position of the orbit, the number of the particles that are passing through the perihelion within the unit of time, shall be constant (independence of the distribution function from the time τ of the perihelion passage); 2. For a certain form of the orbit and for a certain inclination, the particles shall be distributed evenly over the possible perihelion lengths ω and over the possible knot-lengths Ω. The assumption regarding Ω means that a rotation-symmetrical density will be assumed within the interplanetary space (rotation axis through the sun). The assumption regarding ω means that the plane through the sun, which is vertical to the axis of rotation, will have to be the symmetry plane of the density distribution. When this plane does not coincide with the ecliptic-according to C. Hoffmeister (1940), the symmetry plane of the zodiacal light closely follows the orbits of the planets so that it is likely that the invariable plane of the planetary system can be considered, in the first approximation, a symmetry plane - then the inclinations i will have to be calculated from it. As to the assumption of the length of the knots, their distribution in the case of the comets speaks in its favor (cf. J. G. Porter, 1952, p.43). But, it is not sufficient for a perfect representation of reality, particularly inasmuch as the falling of meteors is concerned, since according to G. S. Hawkins (1956), the frequency of the sporadic meteors show, in radar observations, a pronounced seasonal movement even after the effect of the apex movement has been set aside by corrections.

On the basis of the more recent results of meteoric reasearch. we shall be able to limit ourselves to elliptical particle orbits; it will suffice to consider the orbits to be pure Kepler orbits.

2. The distribution function of the orbital elements and the density

With the aid of the suppositions of the preceding paragraphs, we shall be able to clarify the density contribution of the particles with the orbital elements a, e, and i (abbreviated; particles [a,e,i]) in the following way: The orbit [a, e, i, ω, Ω,] is time-independent, according to supposition 1, but it is not occupied evenly by particles. When we vary ω, we shall receive a (doubly occupied) circular ring, within the orbital plane. When we vary Ω, in addition to ω, then we shall obtain (again, doubly spread) a space area (one fourth of the cross section has been shaded in Fig. 1!) that is limited by two rotation cones and by two partial spheres. Exceptions are the orbits with $e \cong 0$ (degeneration of the area into the surface of a spherical zone) and $i \cong 0$ (degeneration into a circular ring). These limiting cases will have to be excluded from the derivation, for the time being.

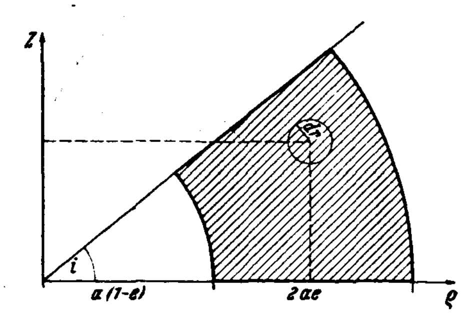

Fig. 1. Illustration of the calculation of the density contribution of the orbits (a, e, i).

Now, the fraction $\frac{dN(\varrho,z)}{N}$ of all particles $[a,e,i]$ will have to be calculated that exists within a torus having a large radius of ϱ and a small radius of d and being located at a distance z from the elliptic (Fig. 1). When one orbit of $[a,e,i,\omega,\Omega]$ passes through ϱ, z, then orbits having a somewhat different perihelion length ω' will also intersect the aforementioned torus. When $\omega + d$ [?] and $\omega - d\omega$ are the perihelion lengths for which the torus will just be touched, then the fraction $\frac{4 \cdot 2 d\omega}{2\pi}$ of all the orbits $[a,e,i]$ will intersect the torus. A factor of 4 will take into consideration that four of such intervals exist for $0 \leqq \omega \leqq 2\pi$ and $0 \leqq \Omega \leqq 2\pi$. Due to the supposition 1 relative to the perihelion passages, the fraction of the particles $[a,e,i,\omega,\Omega]$ in the orbital sector 2 x dr around ϱ, z will be equal to the probability of the location of one single particle in that sector. When the particle requires a time of 2 x dt to pass through the sector, then this probility will amount to 2 x dt/T, when T designates the time of the revolution.

The fraction of the particles on the neighboring orbits within the torus relates to those through ρ, z like the length of the chord to the diameter in the circle. We shall obtain the mean fraction of the particles that are within the torus, by multiplying 2 x dt/T by the ratio of the mean length of the chord and of the diameter in the circle, i.e. with $\pi/4$.

The value sought is the product of the two fractions calculated, hence

$$\frac{dN(\varrho,z)}{N} = \frac{2\,d\omega \cdot dt}{T}. \tag{1}$$

The calculation of $d\omega$ and dt is the problem of the following sections. First, we shall have to compile a few basic equations.

A representation of the orbit $[a,e,i,\omega]$ within the ρ, z-system of coordinates, with the true anomaly V as its parameter, will follow from the known equation for the radius vector, viz.

$$r = \sqrt{\varrho^2 + z^2} = \frac{a(1-e^2)}{1 + e\cos v} \tag{2}$$

and from the equations

$$z = r \cdot \sin i \cdot \sin(\omega + v)$$
$$\varrho = r \cdot \sqrt{1 - \sin^2 i \cdot \sin^2(\omega + v)}\,, \tag{3}$$

which may be read from the rectangular spherical triangle of Fig. 2. The surface equation

$$r^2 \frac{dv}{dt} = \sqrt{G\,\mathfrak{M}_\odot\, a(1 - e^2)} \tag{4}$$

and Kepler's Third Law

$$\frac{2\pi}{T} = \frac{\sqrt{G\,\mathfrak{M}_\odot}}{a^{3/2}}\,. \tag{5}$$

will apply for the connection with the time. As to the length, the Astronomical Unit (A.U.) will be used in the following as unit, while we shall use the year as the unit of time and the mass of the sun, $\mathfrak{M}_\odot$, as the unit of mass. We shall then have to write the gravitation constant as $G = 4\pi^2$.

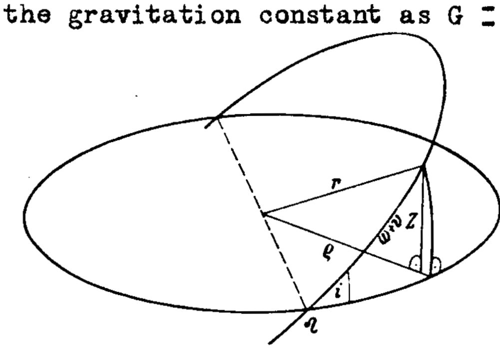

Fig. 2. Connection bewteen the orbital elements and the coordinates ϱ and z.

The calculation of $d\omega$. Let us consider, once more, the point ϱ, z, through which the orbit $[a, e, i, \omega]$ is passing. The change of the perihelion length during the transition to a neighboring point will be

$$d\omega = \frac{\partial \omega}{\partial \varrho}\, d\varrho + \frac{\partial \omega}{\partial z}\, dz\,.$$

In order to find the $d\omega$ of the orbit that touches the circle with dr around ϱ, z, we shall have to choose $d\varrho$ and dz in such a way that the neighboring point will be located on the vertical of the orbit $[a, e, i, \omega]$, and that $d\varrho^2 + dz^2$ will be equal to dr^2. Now, the direction of the orbital tangent will be determined by the vector $\left(\frac{\partial \varrho}{\partial v}, \frac{\partial z}{\partial v}\right)$, while the direction of the vertical will be determined by $\left(-\frac{\partial z}{\partial v}, \frac{\partial \varrho}{\partial v}\right)$.

Hence, we shall have

$$d\varrho = dr\cdot\frac{d\varrho}{\partial v}\cdot\Delta^{-1},\ dz = -\,dr\cdot\frac{dz}{dv}\cdot\Delta^{-1},\ \Delta^2 = \left(\frac{\partial\varrho}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2$$

and-only the amount will be of interest in the following-

$$|d\omega| = |dr|\cdot\Delta^{-1}\cdot\left|\frac{\partial\omega}{\partial\varrho}\cdot\frac{\partial\varrho}{\partial v} - \frac{\partial\omega}{\partial z}\cdot\frac{\partial z}{\partial v}\right|. \tag{6}$$

The further task will be merely a calculating job. We differentiate equation (3) for v after we have eliminated r from it, with the aid of (2). In the result, W and v may be eliminated by expressions in W and z. Then, the first equation of (3) will be solved for ω, v will be eliminated with the aid of equation (2), and $\frac{d\omega}{\partial\varrho}$ as well as $\frac{\partial\omega}{\partial z}$ will be formed, while sin v will be eliminated. Then the expressions will be developed from the partial derivations, with the following result:

$$|d\omega| = \frac{|dr|\cdot\sqrt{2-\frac{r}{a}-\frac{ar}{\varrho^2}(1-e^2)\cos^2 i}}{r\cdot\sqrt{2-\frac{r}{a}-\frac{a}{r}(1-e^2)}\ \sqrt{\sin^2 i-\frac{z^2}{\varrho^2}\cos^2 i}}. \tag{7}$$

<u>Calculation of dt.</u> Since the connection between modifications of the time and of the true anomaly is known by the surface equation, we shall calculate its modification during the passage through the torus. The following will apply:

$$dz^2 + d\varrho^2 = \left(\frac{\partial z}{\partial v}\right)^2 dv^2 + \left(\frac{\partial\varrho}{\partial v}\right)^2 dv^2 = dr^2$$
$$|dv| = |dr|\cdot\Delta^{-1}. \tag{8}$$

The denominator of the right side has already been calculated for $|d\omega|$. It follows then, with the aid of equations (4) and (5), that

$$\frac{|dt|}{T} = \frac{|dr|}{2\pi\, r\left(\frac{a}{r}\right)^{3/2}\sqrt{2-\frac{r}{a}-\frac{ar}{\varrho^2}(1-e^2)\cos^2 i}}. \tag{9}$$

<u>The integral relation for the density,</u> Equation (1) indicates the relative frequency of the particles [a,e,i] in a torus having a cross-section of π (dr)^2 and a circumference of $2\pi\rho$. Hence, we shall obtain the density contribution n*(a,e,i) of this type of particles, in a volume element of d ρ × dz x 1= 1

around the point ρ /illegible/ by multiplying by N(aie,i)and by dividing by $2\pi\varrho\cdot\pi(dr)^2$:

$$n^*(a,e,i;\varrho,z)=\frac{|d\omega|\cdot|dt|\cdot N(a,e,i)}{\pi\varrho\cdot\pi\,|dr|^2\cdot T}. \tag{10}$$

By means of integration over all possible values of the orbital elements, we shall obtain the particle density in the point under consideration, with the aid of equations (7) and (9):

$$\frac{1}{2\pi^3\varrho}\int\limits_a\int\limits_e\int\limits_i\frac{N(a,e,i)\,da\cdot de\cdot di}{r^2\left(\frac{a}{r}\right)^{3/2}\sqrt{2-\frac{r}{a}-\frac{a}{r}(1-e^2)}\;\sqrt{\sin^2 i-\frac{z^2}{\varrho^2}\cos^2 i}}. \tag{11}$$

The possible values of the orbital elements which are evident already from the clear explanation given at the outset, will also result from the roots in Equation (11). The perihelion distance will have to be smaller than r, while the aphelion distance will have to be larger:

$$a(1-e)\leqq r\leqq a(1+e).$$

This assumption will supply, for a and e, the integration range of

$$\mathfrak{B}\begin{cases}\frac{1}{1+e}\leqq\frac{a}{r}\leqq\frac{1}{1-e}\\ 0\leqq e\leqq 1\end{cases} \tag{12}$$

(Fig. 5, p. 24)

Furthermore, only those particles will contribute to the density that are in orbits, the inclination of which is so large that the orbital plane will, at least, touch the torus around ρ, z, i.e., when we shall have

$$\left|\frac{z}{\varrho}\right|\leqq|\operatorname{tg} i|\leqq\frac{\pi}{2} \tag{13}$$

3. Incidence of particles on the earth.

The orbit of the earth will be idealized as a circular orbit having a radius of 1. The radius of the earth, R, will be considered small in relation to all other lengths that will occur. The following formulae will apply directly to other planets also, when we use their large semi-axis as the unit of length and their time of revolution as the unit of time. The reference plane introduced in I, 2 is the orbit of the earth or of a planet.

If we also introduce - besides the coordinates ϱ and z as used up this point - the ecliptical length l, then it is evident on the basis of our earlier considerations that the density contribution of the particles [a,e,i] in one point ϱ,z,l will be supplied solely by two intervals of dω and two intervals of dΩ. This means: In each point of their orbit, the particles with [a,e,i] approach the earth from four discrete directions with a uniform relative velocity of U, since that velocity depends solely on a, e, i. The density of the flow will be determined by n*(a,e,i), in accordance with equation (10). The earth will lay hold of the particles having an "effective cross-section" of πR^{*2}, by which the effect of the attraction of the earth will be taken into account, at least, in the first approximation; for that reason, the cross-section will depend on the relative velocity. Since the year serves as the unit of time, we shall obtain the number of the meteors or micro-meteors with the original elements a, e, and i per day by

$$m^*(a,e,i) = \frac{1}{365}\pi R^{*2}(U)\cdot U(a,e,i)\cdot n^*(a,e,i;1,0). \tag{14}$$

Their total number will be determined by integrating over the orbital elements when, for the integration range in I, we shall have to assume that 2r=1.

<u>The relative velocity</u> may be obtained from the velocity of the circular orbit of the earth, V = 2 A.U./a and from the velocity w of the particle that is assumed to have the components w_ϱ, w_z, w_l. We shall then have

$$U = \sqrt{(V-w_l)^2 + w_\varrho^2 + w_z^2} = \sqrt{V^2 + w^2 - 2Vw_l}. \tag{15}$$

According to the energy equation, we shall find that, at a distance of r= 1, there will be

$$w^2 = 4\pi^2\left(2 - \frac{1}{a}\right).$$

The tangential velocity within the orbital plane is, for r=1, according to equation (4), $1\cdot\frac{dv}{dt} = 2\pi\sqrt{a(1-e^2)}$. The projection of this velocity into the reference plane will be, during the passage through the knot, as follows:

$$w_l = 2\pi\sqrt{a(1-e^2)}\cos i;$$
$$U = 2\pi U' = 2\pi\sqrt{3 - \frac{1}{a} - 2\sqrt{a(1-e^2)}\cos i}. \qquad (16)$$

While the earlier equations were valid independently of the force of attraction of the central mass, the central mass does enter into the relative velocity. For that reason, we shall have to keep in mind that, in the case of small particles, the effect of gravity will be cancelled out, in part, by the radiation pressure. Since both forces have the same law of distance, we shall have to take the radiation pressure into consideration by introducing a reduced solar mass of $\mathfrak{M} = \mathfrak{M}_\odot(1-\delta)$, when δ signifies the ratio of the radiation force and the gravitation force of the sun. For such particles, we shall find, in lieu of equation (16), that

$$U_\delta = 2\pi\sqrt{1 + (1-\delta)\left(2 - \frac{1}{a}\right) - 2\sqrt{a(1-e^2)(1-\delta)}\cos i}. \qquad (17)$$

Numerical data concerning δ as dependent on the radius of the particle for totally-reflecting and metal particles may be found in C. Schalén's paper (1938). Roughly, it is possible to state that δ will assume such values, in the case of radii of the particles that are smaller than 10^{-3} cm, that it cannot be disregarded any longer.

As to the effective catchment cross-section /"Auffangquerschnitt"/ of the earth, we shall obtain an estimate by means of the well-known energy-impulse consideration. It is a rather rough one, since the attraction of the sun is being disregarded. Then, the orbits of the particles near the earth will be hyperbolae having the center of the earth as their center. We shall then try to find the hyperbola that will just be touching the earth by its vertex. When U is the relative velocity of a particle in relation to the earth, at a large distance, then the velocity in the vertex will be

$$v^2 = \frac{8\pi^2\mu}{R} + U^2.$$

according to the energy equation. For the distance R^* of the orbit from the parallel to the velocity vector, at a large distance from the earth, through the center of the earth, i.e., for the "thrust parameter", the impulse theorem will supply the condition that

$$R^* \cdot U = R \cdot v\,.$$

It follows therefore that

$$R^{*2} = R^2\left(1 + \frac{2\mu}{RU'^2}\right). \tag{18}$$

So as to remain within the chosen system of measurements, we shall have to express μ, i.e. the mass of the earth, in units of the mass of the sun, i.e., $\mu = 3.01 \times 10^{-6}$; R is the radius of the earth plus the effective height of the atmosphere (some 130 km). Thereby, the constant within the parentheses will become

$$\frac{2\mu}{R} = 0.1386\,.$$

For $\delta < 1$, the relative velocity (17) will achieve its minimum on the edge of (12): $a = \frac{1}{1+e}\left(\frac{1}{2} \leq a \leq 1\right)$ or $a = \frac{1}{1-e}\ (1 \leq a < \infty)$ for $i = 0^\circ$ - and its maximum, also at that location, for $i = 180^\circ$. There, the following will apply:

$$(U_\delta)_{\text{edge}} = 2\pi\left|1 \mp \sqrt{(1-\delta)\left(2-\frac{1}{a}\right)}\right| \begin{cases} - \text{ Minimum} \\ + \text{ Maximum}. \end{cases} \tag{19}$$

For $\delta = 0$, the relative velocity will be 0 at $a = 1$ while, for $0 \leq \delta \leq \frac{1}{2}$, it will be equal to $\frac{1}{1-\delta}$ at 2, and there only. Then, the approximation for the effective interception cross-section (18) will be useless. An upper limit for the maximal catchment cross-section may be obtained by considering the restricted three-bodies problem.

In the system of sun and earth at rest, the Jacobi integral

$$v^2 = 2\,\Omega(x, y, z) - C\,,$$

when Ω is some sort of a potential, applies to the velocity of a particle having Jacobi's constant C. Hill's limit curves of $2\Omega = C$ restrict the range of movement

The integrand of the last equation is, when we disregard the distribution function and the factor $a^{-3/2}$, Oepik's collision probability per revolution of the particle [1951, Equation (23)]. But, the derivation of the formula as found in this paper shows that the denominator of this equation is exact, provided that the interplanetary density in regions of the order of magnitude of the catchment cross-section of the earth may be considered constant. It will also be possible to write, by using the density, equation (11), that

$$m = \frac{\pi R^2 \cdot n(1,0)}{365} \cdot 2\pi \frac{\displaystyle\iiint \frac{\left(U' + \frac{0.139}{U'}\right) N(a,e,i)\, da\, de\, di}{a^{3/2}\sqrt{2 - \frac{1}{a} - a(1-e^2)}\sin i}}{\displaystyle\iiint \frac{N(a,e,i)\, da\, de\, di}{a^{3/2}\sqrt{2 - \frac{1}{a} - a(1-e^2)}\sin i}} = \frac{\pi R^2 n(1,0)}{365} \cdot 2\pi\, U_{eff}\,. \tag{20}$$

The "effective incidence velocity" U_{eff}, which has been defined thereby, will be of the order of magnitude 1, in accordance with equations (17) and (19), when we disregard the case of $U=0$.

On the basis of equation (20), we shall obtain - from the distribution function of the orbital elements N (a,e,i) - independent estimates of the minimum and maximum numbers of the incident particles; on account of the assumptions made for the derivation, however, these estimates are not altogether cogent. The function over which the mean value is ascertained in (20), will have a minimum of the value of $2U'=0.744$ for $U'=0.1386$. Thereby, we shall obtain a lower limit of

$$m_{min} = 2\pi \cdot 0.744 \cdot \frac{\pi R^2}{365} \cdot n(1,0)\,. \tag{20'}$$

An upper limit can be determined only for particles, in which the radiation pressure is not very considerable. It will then follow from (18') that

$$m_{max} = 2\pi \cdot 265 \cdot \frac{\pi R^2}{365} \cdot n(1,0)\,. \tag{20''}$$

The upper limit presented here is rather unlikely, but - without any assumptions regarding the model - it can be depressed only, to some extent, by more precise calculations in the field of celestial mechanics.

It is possible that, due to the secular gathering of dust by the earth that has been discussed in detail by E. Oepik (1951), a discontinuity of the dust distribution of the interplanetary dust prevails at the distance of the earth from the sun. This would have to be particularly noticeable in the case of the larger particles, in which such an effect will be less obscured by disturbances. Such a discontinuity will go unnoticed in zodiacal light observations, since there, the mean values are always determined over large areas. The conclusions relating to the distribution function of the orbital elements as obtained on the basis of the integral for the particle incidence (20) are, therefore, not so generally valid as the ones reached on the basis of the density integral (11).

II. A model for the distribution function of the orbital elements.

1. Conditions. First conclusions.

We shall consider models that are as simple as possible, in order to obtain an idea of the connection of density and distribution function as formulated in equation (11). The distribution function shall be the product of three functions each one of which shall depend on one orbital element only:

$$N(a,e,i) = N_a \cdot N_e \cdot N_i \,. \tag{21}$$

Hence, the integration over i can be separated directly from the two other ones. This first integral,

$$n_i = \int\limits_{\operatorname{arctg} z/\varrho}^{\pi/2} \frac{N_i \, di}{\sqrt{\sin^2 i - \frac{z^2}{\varrho^2}\cos^2 i}} \tag{22}$$

depends solely on z/ϱ. As to the density problem, the sense of the rotation of the particles is - as may also be seen from the denominator of the integrand - without any importance. For that reason, the sum of the direct orbits of the inclination i and of the retrograde orbits having an inclination of 180° - i will be designated as N_i, and the integration will be extended solely over one half of the interval (13).

July 24, 1965

To: Mr. John M. Weaver, Librarian
Goddard Space Flight Center, NASA

Ref: Contract No. NAS 5-8010, Item 43

From: Franklin W. Clark

I inclose the following German translation:

Haug: "On the frequency distribution of the orbital elements in the inter-planetary dust particles" [From: Zeitschrift für Astrophysik, 44: 71-97 (1958)]

There are approximately 9,900 words.

Best regards.

With the aid of $x = a/r$, we shall obtain the two other integrals in the form of

$$n_{a,e} = \frac{1}{\varrho \cdot r} \int\limits_{e=0}^{1} \int\limits_{x=\frac{1}{1+e}}^{\frac{1}{1-e}} \frac{N_a(x \cdot r) \cdot N_e \, dx \, de}{x\sqrt{2x - 1 - x^2(1-e^2)}} . \tag{23}$$

In the ecliptic, $\varrho = r$ and $n_{a,e} = n_{a,e}(\varrho)$ will apply, and the density distribution will be - when we assume the convergence of the double integral -

$$(n_{a,e})_{Ekl.} = c \cdot \varrho^{k-2} \quad \text{for} \quad N_a = c' a^k . \tag{24}$$

For that reason, we shall be able to represent by formula (21) a prescribed density distribution in the ecliptic and at a certain distance from the sun and vertically to it, by a suitable choice of the distribution function. This agrees with the data that may be determined, e.g., on the basis of zodiacal light observations. Here, N_e will still remain undetermined, except for convergence conditions. Clues for its selection will result from considerations of the supposed dust generators in the planetary system or - as we shall explain in more detail in Section II, 4 - from the relative frequency of the eccentricities in meteors. It follows from equations (22) and (23) that the density amounts to

$$n(\varrho, z) = n_{a,e}(\varrho, z) \cdot n_i\left(\frac{z}{\varrho}\right). \tag{25}$$

2. Discussion of the exponential laws governing the distribution of large semi-axes.

We shall consider here but a few whole values of k. Intermediate values are possible, but in that case, we shall have to determine numerically the value of the double integral in (23) for an appropriate N_e.

$k = 0$, i.e. - in accordance with equation (24) - a density decrease with ϱ^{-2} will hardly be assumed any longer today. In that case, there would be

$$n_{a,e} = \frac{1}{\varrho \cdot r} \int\limits_{e=0}^{1} N_e \left[\arcsin \frac{x-1}{e\,x}\right]_{\frac{1}{1+e}}^{\frac{1}{1-e}} de = \frac{\pi}{\varrho \cdot r} \int\limits_{0}^{1} N_e \, de . \tag{23a}$$

For N_e const. the density contribution of equal e-intervals would be equal. For the more probable case of N_e toward 0 for e toward 1, the density contribution of the particles of smaller eccentricity would be larger. When we first integrate over e (for N_e = const), then we shall find that the density contribution of the particles with $1/2 \leq a \leq 1$ would have to be equal to the particle with $1 \leq a < \infty$. Since the integral

$$\iint \frac{dx\,de}{x\sqrt{2x-1-x^2(1-e^2)}}$$

can be indicated also for regions forming part of $\mathfrak{B}$, it will be suitable to serve as the basis of numerical calculations, in which N_a and N_e will be considered to be approximately constant in those regions.

A density decrease with, approximately, ϱ^{-1} will result when it is desired to explain the zodiacal light solely on the basis of the scattering of the light of the sun. If the polarization of the zodiacal light is a consequence of the scattering on free electrons, then there will remain a share of luminosity that, near the earth, will best be represented by a constant dust density in the ecliptic. This density will be, for $k = 2$.

$$n_{a,e} = \frac{r}{\varrho} \int\limits_{e=0}^{1} N_e \left[\int\limits_{\frac{1}{1+e}}^{\frac{1}{1-e}} \frac{x\,dx}{\sqrt{2x-1-x^2(1-e^2)}} \right] de = \frac{\pi r}{\varrho} \int\limits_{e=0}^{1} \frac{N_e\,de}{(1-e^2)^{3/2}}. \tag{23b}$$

N_e must, for $e \longrightarrow 1$, move more pronouncedly toward 0 than $(1-e)^{\frac{1}{2}}$. There will be

$$n_{a,e} = \begin{cases} \dfrac{\pi^2}{2} \cdot \dfrac{r}{\varrho} & \text{for} \quad N_e = 1 - e^2, \\ \pi \cdot \dfrac{r}{\varrho} & \text{for} \quad N_e = (1-e^2)^{3/2}. \end{cases}$$

In the first case, the density contribution will increase as the excentricity increases, while it will be constant in the second case. Since, undoubtedly, an essential part of the dust is being generated by comets with a larger e, a law of distribution involving the smallest possible decrease, as e increases, is more probable. In the following, we shall consider the distribution law $N_a \times N_e = a^2(1-e^2)$ exclusively.

Since, in the case of this law of distribution, the contribution of large values of a to the density is very considerable, we shall have to examine which effect an ending of the dust cloud will have at a maximal large semi-axis or aphelion distance.

It follows from the numerical calculations in II, 4 that the double integral in (23b), and thereby the density, will decrease between $\varrho = 0.5$ and $\varrho = 1.5$ by 10% in the case of $a_{max} = 20$ A.U., and by 12% in the case of $a_{max} = 10$ A.U. (When we assume maximal aphelion distances, then the same values will result for $[a(1+e)]_{max} = 20$ A.U. and $[a(1+e)]_{max} = 40$ A.U., respectively). This decrease of the density will partially be cancelled out again by the absence of the particles in which a is smaller than a minimal large semi-axis, or in which a(1-e) is smaller than a minimal perihelion distance. Since the vaporization limit is largely dependent on the physical properties of the particles, we made only an estimate in the same manner as above, for the purpose of preliminary orientation, to the effect that an absence of all particles with $a < 0.4$ would bring about a density increase of 5% between $\varrho = 0.5$ and $\varrho = 1.0$. A minimal perihelion distance of 0.1 A.U. (cf. van de Hulst, 1947, p. 480) will be but slightly noticeable according to the law of distribution adopted: between $\varrho = 0.5$ and $\varrho = 1.5$ A.U., we shall have an increase in density of less than 4%.

It follows from equations (22) and (23b) that, for k 2, the density will depend solely on the ratio ϱ/r. When we consider this from the point of view of space, then the areas of constant density will be cones the axis of which is vertical to the ecliptic and the vertex of which coincides with the sun. Such a distribution of density appears rather plausible for particles on Kepler orbits within the field of a central mass. But, as it happens, the density decrease that is vertical to the ecliptic has been derived by Elsasser (1954, p. 281) on the assumption that the surfaces of constant density are planes that are parallel to the ecliptic. When we plot the curves of constant density according to the two assumptions in the elongations of the sun as applied ($\varphi = 35^0$, 42.5^0, 65^0, 80^0) then it will become apparent that, on the basis of the new assumption, the lengths of the visual rays will be reduced in the regions of higher density. So as to be able to represent the measured values of luminosity now, just as before, we should have to assume that the density at the distance of the earth from the sun decreases more slowly in a vertical direction to the ecliptic than on the earlier assumption. But, we shall be able to see, at the same time, that - for all the elongations in the ecliptic used and for all the inclinations toward the ecliptic applied - the effect for $z = 0.25$ A.U. will not exceed 10%, and that it also will not exceed 50% for the larger distances. But, the data obtained so far are not any more precise than that, as has been shown already by the difference of the curves I and II that show the decrease in density, as presented by Elsasser. For that reason, these curves will be used unchanged in the derivation of the distribution of the inclination. But, it would be desirable to have new observations reduced, on the assumption that the cones under consideration here are surfaces of constant density.

3. Determination of the distribution of the inclination

For this purpose, we used two methods: on the one hand, we evaluated equation (22) for various models of N_i; on the other hand, we solved the integral equation for N_i by approximation on the basis of the empirical density distributions.

When the inclinations are distributed evenly, we find - as it is well known - that

$$N_i = 2\sin i; \tag{A}$$

$$n_i = \frac{2\varrho}{r}\int_{\arccos \varrho/r}^{\pi/2} \frac{\sin i\, di}{\sqrt{\frac{\varrho^2}{r^2} - \cos^2 i}} = \frac{2\varrho}{r}\left[\arccos \frac{r\cos i}{\varrho}\right]_{\arccos \varrho/r}^{\pi/2} = \frac{\pi\varrho}{r}. \tag{22a}$$

This distribution is intended to serve as a reference function when we resolve the integral into a sum. If we simply pull out N_i, in a partial interval, from the integral (22), then we shall encounter convergence difficulties for $i = 0$, since each distribution is to have a maximum, but not a peak, for $z = 0$.

When the density decreases vertically to the ecliptic, then N_i will have to move toward 0 for $i \longrightarrow \frac{\pi}{2}$. This will, e.g., be the case at cos i. This will lead to models having a form of

$$N_i = c \cdot \sin i \cdot \cos^{\nu} i. \tag{B}$$

due to the increase with sin i for small i's. Then we shall find that

$$n_i = \frac{c\varrho}{r}\int_0^{\varrho/r} \frac{t^{\nu} dt}{\sqrt{\frac{\varrho^2}{r^2} - t^2}} = \left(\frac{\varrho}{r}\right)^{\nu}, \tag{22b}$$

when N_i is fixed in such a way that $n_i(0) = 1$. High values of ν will be required to represent the empirical curves. This characterizes the strong deviation of the empirical distribution of the inclinations from the equipartition. Fig. 3 shows the n_i's as taken from Elsasser's curves I and II (1954, p. 284; I corresponds to the decrease of luminosity North of the ecliptic; II corresponds to it South of the ecliptic) for ν = 24, 48, and 64.

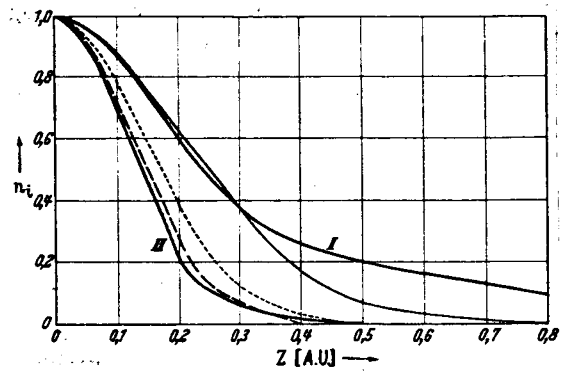

Fig. 3. Distribution n_i on the basis of the density decrease taking place vertically to the ecliptic, according to zodiacal light observations. (H. Elsasser, 1954; curves I and II), and according to Model B for = 24 (———), = 48 (- - - -) and = 64 (— — —)

A closed presentation may be desirable for some theoretical problems - it is also possible still to improve the models B without any difficulty - but when an exact approximation of the empirical curves is of decisive importance, then an inversion of the integral equation (22) for N_i will be preferable. This inversion can, as may be mentioned parenthetically, be carried out in an exact manner. By means of transformation, we shall obtain an Abelian integral equation for $N_i/\sin_i$, with the inversion

$$\frac{N_i}{\sin i} = -\frac{2}{\pi} \int_{\operatorname{tg} i}^{\infty} \frac{\left[(1+s^2)\frac{d n_i(\varrho \cdot s)}{ds} + s\, n_i(\varrho \cdot s)\right] ds}{\sqrt{\frac{s^2}{1+s^2} - \operatorname{tg}^2 i}} . \tag{26}$$

For the purpose of practical calculation, it is more convenient to resolve the integral (22) into a sum, by dividing the i-region into K ≠ 1 intervals. Now, however, not N_i but - as intimated before - $N_i/\sin_i$ shall be considered constant for $i \to 0$, because of the behavior of N_i. Then, we shall have

$$n_i(z_l) = \sum_{k=0}^{K} \left(\frac{N_i}{\sin i}\right)_k \cdot J_{k,l} . \tag{27}$$

For the coefficients, we shall find with the aid of equation (22a) that

$$J_{k,l} = \begin{cases} \dfrac{\varrho}{\sqrt{\varrho^2 + z_l^2}} \left(\arccos \dfrac{\sqrt{\varrho^2 + z_l^2} \cdot \cos i_k}{\varrho} - \arccos \dfrac{\sqrt{\varrho^2 + z_l^2}\,\cos i_{k+1}}{\varrho} \right) \\ \qquad \text{for otherwise} \arccos \dfrac{\varrho}{\sqrt{\varrho^2 + z_l^2}} \leqq i_k \leqq \dfrac{\pi}{2} \\ 0 \end{cases} \tag{28}$$

In the case of a definite i-division, we shall select the abscissae z_l for the reference points of the curves of the density decrease in such a way that

$$\arccos \frac{\varrho}{\sqrt{\varrho^2 + z_l^2}} = i_l \qquad l = 0, 1, \ldots, K$$

Then, there will be

$$\left(\frac{N_i}{\sin i}\right)_K = \frac{n_i(z_K)}{J_{K,K}}$$

and generally

$$\left(\frac{N_i}{\sin i}\right)_l = J_{l,l}^{-1} \left\{ n_i(z_l) - \sum_{k=l+1}^{K} \left(\frac{N_i}{\sin i}\right)_k \cdot J_{k,l} \right\}, \tag{27?}$$

i.e., it will be possible to resolve the integral equation from large z's. On the basis of the values found, we shall obtain $(N_i)_l$ by multiplying by

$$(\sin i)_l = \frac{\cos i_l - \cos i_{l+1}}{i_{l+1} - i_l}.$$

Intervals having a latitide of 5° were selected for the evaluation of Elsasser's curves for $0 \leqq i \leqq 4$[illegible] while intervals of a latitude of 15° were selected for $45 \leqq i \leqq 90°$, i.e., for z values for which the curves of the density decrease will have to be extrapolated. Table I shows the values of $J_{k,l}$, while Table 2 presents the found values for $(N_i/\sin i)_l$ J_t and $(N_i)_l$ for Elsasser's curves I and II and for the mean III of these two curves. In Fig. 4, the percentage-wise frequencies of the inclinations have been compared with the interplanetary frequencies of sporadic meteors as found by F. L. Whipple (1954, p. 216, Table V, column 4). The agreement - particularly of I - with the data relating to the meteors is surprisingly good.

4. Calculation of the relative frequency of the orbital elements of meteors.

We shall now calculate the relative frequencies of the orbital elements a, and e, and i, in the case of (micro-) meteors, so as to arrive at a first comparison of

Table 1. Coefficients [illegible] for the approximative solution of integral equation (22) for the inclination distribution

l	z/ϱ \ k	1	2	3	4	5	6	7	8	9	10	11	12
	$i°$	0—5	5—10	10—15	15—20	20—25	25—30	30—35	35—40	40—45	45—60	60—75	75—90
1	.0000	.0873	.0873	.0873	.0873	.0873	.0873	.0873	.0873	.0873	.2618	.2618	.2618
2	.0875		.1507	.0956	.0908	.0895	.0883	.0881	.0881	.0875	.2622	.2620	.2618
3	.1763			.1934	.1057	.0968	.0928	.0908	.0897	.0889	.2644	.2630	.2618
4	.2679				.2254	.1158	.1020	.0961	.0936	.0910	.2680	.2635	.2618
5	.3640					.2514	.1230	.1066	.0996	.0951	.2731	.2652	.2621
6	.4663						.2716	.1289	.1107	.1012	.2819	.2668	.2626
7	.5774							.2804	.1345	.1126	.2943	.2702	.2627
8	.7002								.3139	.1365	.3156	.2745	.2634
9	.8391									.3024	.3560	.2808	.2640
10	1.0000										.5553	.2904	.2650
11	1.732											.5105	.2720
12	3.732												.4065

Table 2. Factors J_l and the inclination distribution on the basis of the density decrease taking place vertically to the ecliptic.

	l	1	2	3	4	5	6	7	8	9	10	11	12
	$i°$	0—5	5—10	10—15	15—20	20—25	25—30	30—35	35—40	40—45	45—60	60—75	75—90
J_l	I	3.533	2.764	1.970	1.907	.667	.499	.310	.229	.177	.067	.010	.000
	II	5.576	4.172	1.158	.344	.116	.057	.036	.000	.000	.000	.000	.000
	III	4.522	3.580	1.586	.722	.329	.250	.157	.114	.090	.036	.000	.000
$(N_i)_l$	I	.154	.361	.424	.324	.256	.230	.166	.140	.116	.053	.009	.000
	II	.242	.545	.294	.103	.044	.026	.019	.000	.000	.000	.000	.000
	III	.197	.467	.342	.217	.126	.115	.084	.069	.061	.028	.000	.000

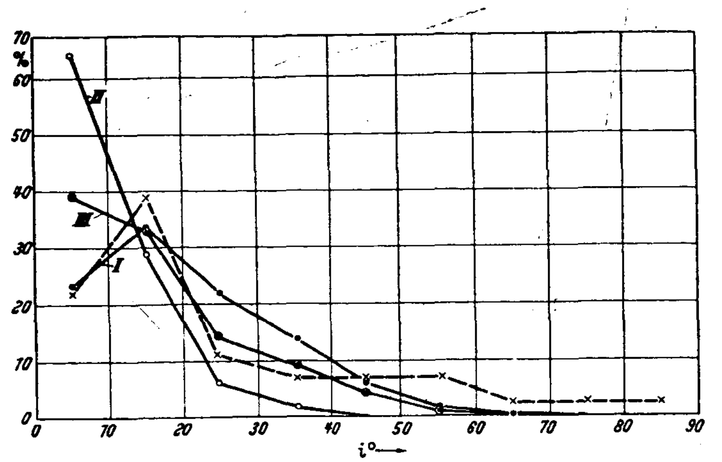

Fig. 4. Percentage-wise frequency of the inclinations in intervals having a latitude of 10° for the share of dust of the zodiacal light (curves I to III) and in the case of meteors (interplanetary), according to F. L. Whipple (1954; -x-)

the consequences of the model found with the material regarding the orbits of meteors; very little material of this type has been published up to this time. The photographic results obtained by the use of the Super-Schmidt cameras [more than 4,000 two-station photographs of meteors up to the beginning of 1956 (cf. D. H. Menzel, 1955)] and the results of the three-station observations by radar [2,400 sporadic meteors (cf. A.C.B.Lovell, 1956)] will soon make a comparison with small regions da x de x di of the orbital elements, according to equation (14) possible.

Since the closed integration of equation (20) fails, on account of the complicated connection of the orbital elements in U, the integration region $\mathfrak{B}$ of a and e was subdivided into 52 partial regions. Fig. 5 shows this subdivision (43 regions) for $a \leqq 5$ A.U.; for larger a's, the strip between $a = 1/1-e$ and $e = 1$ up to 10 A.U. was subdivided into pieces having a longitude of 1 A.U.; it was further subdivided at a = 15, 20, and 100 A.U. (9 regions). We calculated

$$A_\nu = \iint\limits_{\Delta\mathfrak{B}_\nu} \frac{da\,de}{a\sqrt{2a-1-a^2(1-e^2)}} \tag{29}$$

for each region $\Delta\mathfrak{B}_\nu$. In addition, we determined the frequency factors of the model being considered, viz, a_ν^2 and $1 - e_\nu^2$, and finally the products $A_\nu\, a_\nu^2\, (1 - e_\nu^2)$ for the points of gravity $a_\nu\, e$ of the surfaces $\Delta\mathfrak{B}_\nu$. These factors have been indicated in Table 3, together with the sums of the products for constant e [sum (a)] and constant e [sum (e)]. The Table affords, by means of a comparison of the individual values with the sum of all the products 4.2533, an idea of the contribution of the partial regions $\Delta\mathfrak{B}_\nu$, to the interplanetary density in the cause of the model under consideration. Finally, we calculated at the same locations a_ν and e_ν, and for $i_\mu = 5(2\mu - 1)(\mu = 1, ... , 9)$, and in accordance with equation (15), the relative velocities $U'_{\nu,\mu}$ in units of the velocity of the orbit of the earth, and the "effective" relative velocities $(U_{\nu\mu})_{eff} = U'_{\nu\mu} + 0.139/U'_{\nu\mu}$. Table 4 indicates the values of $(U_{\nu\mu})_{eff}$ for $i = 5^\circ$; all of them are located between 0.7 and 1.3. It is true that this is due, in part, to the gross subdivision into intervals. In the following, we shall disregard the retrograde orbits which are considerably less frequent among the orbits the meteors, despite a greater probability of impingement. The frequency factors of the inclination in the interval around i_μ are

$$J_\mu = \int_{i_\mu - 5}^{i_\mu + 5} \frac{N_i}{\sin i}\, di . \tag{30}$$

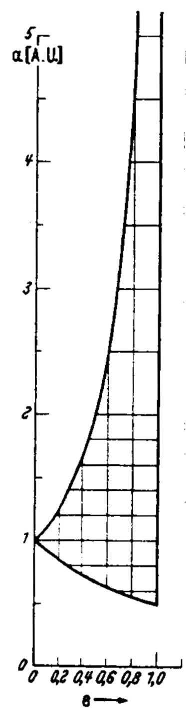

Fig. 5. Integration region $\mathfrak{B}$ of the orbital elements a and e and its subdivision for the purpose of numerical integration.

In the calculation, we applied the values I in Table 2 to that equation. Then, we shall have double sums instead of the triple integrals in equation (20), and the "effective velocity of incidence" as defined there will become

$$U_{eff} = \frac{\sum_{\mu=1}^{9} \sum_{\nu=1}^{52} (U_{\nu\mu})_{eff} \cdot J_\mu A_\nu a_\nu^2 (1 - e_\nu^2)}{\sum_{\mu=1}^{9} J_\mu \sum_{\nu=1}^{52} A_\nu a_\nu^2 (1 - e_\nu^2)} . \tag{31}$$

When we evaluate the sums in the numerator in part only, and when we divide by the total sum (31), then we shall have the relative frequency of the path elements in meteors. E.g., the frequency of the inclinations will turn out to be, when we do not sum over μ :

$$m_\mu = \frac{\sum_{\nu=1}^{52} (U_{\nu\mu})_{eff} \cdot J_\mu A_\nu a_\nu^2 (1 - e_\nu^2)}{\sum_{\mu=1}^{9} \sum_{\nu=1}^{52} (U_{\nu\mu})_{eff} \cdot J_\mu A_\nu a_\nu^2 (1 - e_\nu^2)} \qquad \mu = 1, \ldots, 9.$$

In the same way, the distribution of the large semi-axes and the sum of the eccentricities may be obtained by summing at ν only over those series that contain all partial regions of a certain e-interval or a-interval.

Table 3. Coefficients $A_\nu a_\nu^2 (1 - e_\nu^2)$

a [A. U.] \ e	0.0—0.2	0.2—0.4	0.4—0.6	0.6—0.8	0.8—1.0	Sum (b)
50 —100					.1544	.1544
20 — 50					.2707	.2707
15 — 20					.0983	.0983
10 — 15					.1727	.1727
9 — 10					.0497	.0497
8 — 9					.0598	.0598
7 — 8					.0709	.0709
6 — 7					.0879	.0879
5 — 6					.1173	.1173
4.5— 5.0				.0274	.0539	.0813
4.0— 4.5				.0577	.0435	.1012
3.5— 4.0				.0920	.0385	.1305
3.0— 3.5				.1190	.0346	.1436
2.5— 3.0				.1723	.0306	.2029
2.0— 2.5			.1270	.1340	.0274	.2884
1.8— 2.0			.1022	.0407	.0101	.1530
1.6— 1.8		.0129	.1252	.0367	.0097	.1845
1.4— 1.6		.1071	.0865	.0339	.0091	.2366
1.2— 1.4	.0187	.1905	.0698	.0313	.0088	.3191
1.0— 1.2	.3369	.1374	.0626	.0299	.0086	.5754
0.8— 1.0	.2745	.1616	.0648	.0304	.0087	.5400
0.6— 0.8		.0503	.0820	.0469	.0104	.1896
0.5— 0.6				.0079	.0076	.0155
Sum (a)	.6301	.6598	.7201	.8601	1.3832	4.2533

So as to be precise, we may not compare these calculated frequencies directly with the observed ones, since the discovery probability is another factor going into the latter ones; they may be represented, according to F. L. Whipple (1954, p. 209 sq.), as a function of the velocity of the meteor at the time of its entry into the atmosphere. For the purpose of a first rough comparison, we have done without this correction.

In a paper by F. G. Watson (1959), perhelion distances and inclinations have been derived from meteor radiants, according to a graphic method, on the assumption of parabolic paths. So as to be able to use for our comparison also this numerically larger material which, however, is only provisionally conclusive, due to the assumptions made, we have attempted to calculate the contribution of regions of constant perihelion distance to the interplanetary density, since the frequency distributions that are obtained in this way, are not different to any considerable extent from those occurring during the fall of a meteor; cf. the considerations relating to the "velocity effect" in I, 3).

Table 4. The "effective" relative velocities $(U_{v\mu})_{eff}$ in partial regions of $\Delta\mathfrak{B}$, for $i = 5^{\circ}$ (by way of example).

a [A. U.] \ e	0.0—0.2	0.2—0.4	0.4—0.6	0.6—0.8	0.8—1.0
50 —100					1.132
20 — 50					1.111
15 — 20					1.120
10 — 15					1.111
9 — 10					1.105
8 — 9					1.105
7 — 8					1.097
6 — 7					1.092
5 — 6					1.071
4.5— 5.0				.747	1.093
4.0— 4.5				.756	1.127
3.5— 4.0				.782	1.160
3.0— 3.5				.802	1.192
2.5— 3.0				.818	1.121
2.0— 2.5			.745	.863	1.243
1.8— 2.0			.746	.910	1.253
1.6— 1.8		.803	.749	.932	1.253
1.4— 1.6		.798	.764	.948	1.250
1.2— 1.4	1.048	.776	.783	.968	1.238
1.0— 1.2	1.196	.755	.792	.953	1.214
0.8— 1.0	1.124	.755	.777	.922	1.166
0.6— 0.8		.808	.748	.864	1.078
0.5— 0.6				.796	.961

We shall have to insert the distribution of the inclination $N_e = 1-e^2$ into (23b) and to make the substitution

$$a(1+e) = q; \quad a(1-e) = p$$

or

$$a = \frac{q+p}{2}; \; e = \frac{q-p}{q+p}; \; \begin{vmatrix} \frac{\partial a}{\partial q} & \frac{\partial a}{\partial p} \\ \frac{\partial e}{\partial q} & \frac{\partial e}{\partial p} \end{vmatrix} = \frac{1}{p+q} \tag{32}$$

When we do, the integral region $\mathfrak{B}$ will become the region $\mathfrak{B}'$

$$0 \leqq p \leqq 1; \quad 1 \leqq q < +\infty.$$

Hence:

$$n_{a,e} = n_{p,q} = 2 \int_{p=0}^{1} \int_{q=1}^{\infty} \frac{p \cdot q \, dp \, dq}{(p+q)^2 \sqrt{(1-p)(q-1)}} = \pi \int_0^1 \frac{p(2+p)\,dp}{(p+1)^{3/2} \sqrt{1-p}}. \tag{33}$$

The execution of the second integration will give - as in (23b) - $n_{a,e} = \frac{\pi^2}{2}$.
The contribution of the interval of the perihelion distance of $0 \leqq p_1 \leqq p \leqq p_2 \leqq 1$ will be, in this equation:

$$\Delta n = \pi \left[\arccos p + p \sqrt{\frac{1-p}{1+p}} \right]_{p_2}^{p_1}. \tag{34}$$

The density contribution will increase as the perihelion distance increases. Inasmuch as the frequency distribution of the meteors is concerned, the effect of the velocity counteracts this increase with p, since the velocity will decrease as p increases, at least in the case of the larger aphelion distances and of inclinations that are not overly large $\left(q \geqq \sqrt[3]{2 \cos^2 i}\right)$, as may easily be seen by a consideration of the relative velocity according to equation (16):

$$U = 2\pi \sqrt{3 - \frac{2}{p+q} - 2 \sqrt{\frac{2pq}{p+q}} \cos i} \tag{35}$$

In Figs. 6 to 9, the curves calculated have been compared with the empirical data. In the calculation of the percentage-wise frequencies, we have used the number of $i \leqq 90°$ as that of the meteors for the total frequencies of the inclinations; for the large semi-axes, we used the number of $a \leqq 100$ A.U.. As to the latter ones, we also increased the lengths of the intervals as /illegible/ increased and we calculated the mean meteor frequency per A.U. semi-axis interval by dividing by the length of the interval. On the other hand, we added, for the statistics of the eccentricities the hyperbolic meteors to the interval of $0.8 \leqq e \leqq 1.0$, since the eccentricities originate - according to Whipple - solely in the scattering within the measurements of the velocity. Since the particles that occur in flows also contribute to the zodiacal light, a compromise was used for the statistics: the strong meteoric showers that were - in Whipple's paper - represented by more than 5 single meteors (geminids, perseids, North- and South-taurids, leonids) were in each case counted as two meteors only; no difference was made between meteors originating in weaker showers and sporadic meteors.

When we take into account that the model distribution was established for particles of an entirely different order of magnitude ($r \approx 10^{-3}$ cm)(it is true that, in the choice of the distribution of the eccentricities, the experience gathered in the astronomy of the meteors and of the comets were a determining factor), then it is surprising how similar the curves are. When, by way of a test, we assume that the suppositions that were made for the purpose of evaluating the data relating to the zodiacal light, i.e., particularly the validity of equation (21) are correct, then the following should be stated, in detail, inasmuch as the Figures are concerned:

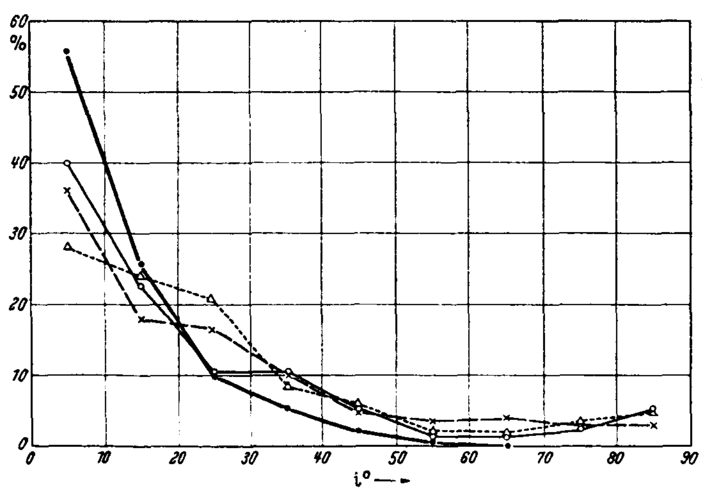

Fig. 6. Percentage-wise frequency of the inclinations of the orbits in the case of meteors, in intervals having a latitude of 10° as calculated on the basis of the model of the dust share of the zodiacal light (-●-), according to the photographic meteor orbits by F. L. Whipple (1954; -o-), and according to meteor radiants by E. Oepik (---- Δ ----) and by Niessl-Hoffmeister (- x -); cf. F. G. Watson, 1939).

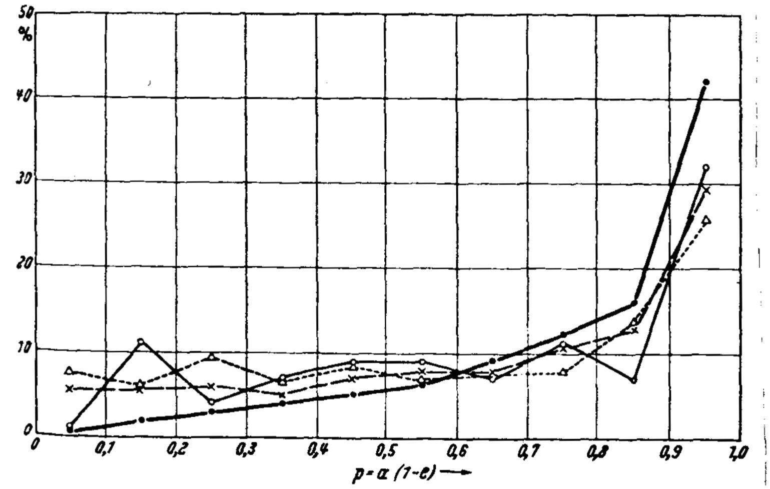

Fig. 7. Frequency of the perihelion distances in the case of meteors: —— Percent by intervals of 0.1 A.U. -- According to F. L. Whipple (1954; -o-), E. Oepik (---- Δ ----), and von Niessl-Hoffmeister (-x-); cf. F. G. Watson, 1939), and interplanetary frequency of the perihelion distances on the basis of the model for the zodiacal light dust (-•-).

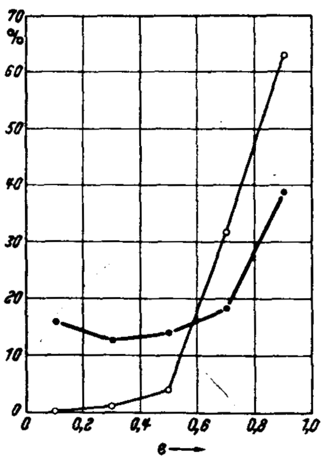

Fig. 8. Percentage-wise frequency of the eccentricities e in the case of meteors, in intervals of 0.2 in e, as calculated on the basis of the model for the dust share of the zodiacal light (-•-), and according to F. L. Whipple (1954; -o-).

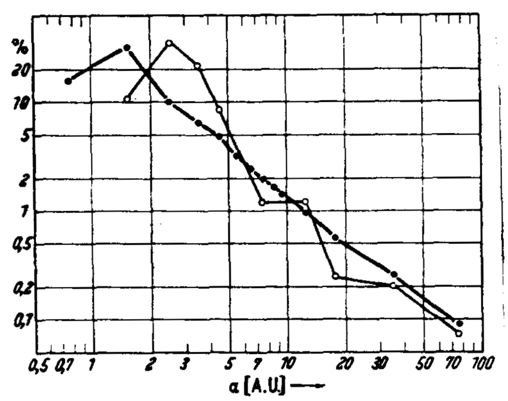

Fig. 9. Percentage-wise frequency of the large semi-axes per interval of 1 A.U., in the case of meteors as calculated on the basis of the model for the dust share of the zodiacal light (-•-), and according to F. L. Whipple (1954; -o-).

Fig. 6. The calculated frequency of small inclinations appears to be somewhat too high. But, it is due less to the steepness of the slope in the case of smaller i's than to the absence of inclinations of > 45° in the theoretical curve. If we analysed the decrease of density, we should obtain these inclinations, e.g., by a

superimposed constant density (inclinations of $\sim$ sin i). The distribution of the inclinations of the comets with long periods (cf. J. G. Porter, 1952, p. 43) suggests that an isotropic part of the inclinations exists. It is very difficult to observe the luminosity effect of such a dust particle in the zodiacal light.

Fig. 7. The correction of the velocity as considered above acts in such a way that the deviations existing between theoretical and empirical distribution will be reduced.

Figs. 8 and 9. Nothing but Whipple's material is available for the comparison of the calculated frequencies for a and e. But, while the curves for large a's and e's are very similar, empirically and theoretically, characteristic differences occur for e $\gtreqqless$ 0.5 and for $0.5 \lesseqqgtr a \lesseqqgtr 2$. In both cases, the consideration of the probability of discovery in the observed curves would have to have the effect of decreasing the deviation. A remaining deviation could be corrected by a modification of the law of distribution of N_e - e.g., $e^2(1-e^2)$ instead of $1-e^2$ - since after all, this law remained indefinite in II, 2. But, according to A.C.B. Lovell (1956), one fourth of all orbits of meteors of $7^m - 8^m$ have, on the basis of radar observations made at the Jodrell Bank Experimental Station, eccentricities of < 0.5 - in the model, the result is approximately 1/3 ! On the basis of the ideas relating to the subsequent delivery /Nachlieferung/ of the dust caught by the sun and the planets as discussed by F. L. Whipple (1955) in detail, it is also probable that the secular gathering effect which we have mentioned several times, has a stronger effect in the case of larger particles, so that - under certain circumstances - the empirical distributions may, in the case of smaller particles, come close to the calculated ones. For that reason, the chosen law of distribution of N_e was retained.

5. Particle incidence and mass incidence upon the earth.

The calculations performed make no new contribution to this problem. The well-known statement (H. C. van de Hulst, 1947; H. K. Kallmann, 1955; M. Minnaert, 1955; F. L. Whipple, 1955) to the effect that interplanetary density is, on the one hand, large enough to result in the expected mass incidence but that, on the other hand, no continuous transition seems to be possible from the interplanetary law of the distribution of radii to the law of the distribution of radii as based on meteor frequencies, remains valid. Inasmuch as the order of magnitude of the radius is concerned, for which this transition might be anticipated, the frequency values as obtained on the basis of the different laws differ by several powers of ten.

In any case, the model established supplies a very definite value of the effective incidence velocity, viz.

$$U_{eff} = 0.95\ .$$

For a model with N_e=const. and a $\leq$20 A.U., we found that U_{eff} is equal to 1.1 We shall now be able to use equation (31) to calculate the particle incidence or the mass incidence - depending on the insertion of the interplanetary particle number/cm^3, or of the interplanetary density, for n(1.0).

According to H. Elsässer (1955) the best simultaneous representation of the Fraunhofer corona and of the dust share of the zodiacal light may be obtained by the following law of distribution of the radii s in the interplanetary dust:

$$n(s)\, ds = 10^{17.86} s^{-2} ds\ [\mathrm{cm}^{-3}]\ .$$

Accordingly, the minimum particle size is located between 10^{-4} and 10^{-3} cm, while the maximal particle size is located between 10^{-2} and 10^{-1}cm. On the basis of a similar law of distribution of the radii, we shall also be able to understand - in accordance with calculations by H. Walter (1957) - the anti-zodiacal light according to Mie's theory. At this moment, the constant of the law is probably the most uncertain point. When we assume the independence of the law of distribution of the orbital elements having the size of particles (section I,1), then we shall find, with the aid of equation (20), in regard to those particles that are larger than 10^{-3}cm U_{eff} is uncertain for smaller particles, on account of the effect of the radiation pressure (I,3)!) that the frequency of micro-meteors is as follows:

$$m(s)\, ds = \frac{\pi R^2}{365} \cdot 2\pi\, U_{eff}\, n(s)\, ds = 10^{29.51-17.86}\, s^{-2}\, ds\ [d^{-1}]\ . \tag{36}$$

Nothing definite can be stated as to the total number of the micro-meteors, since the small particles are the determining factor in that respect. On the other hand, the incident mass depends on the upper limit of the particles, up to which equation (36) will be valid. The total mass will be

$$M \approx 10^{9.65} \cdot \frac{2\pi\sigma}{3} s_{max}^2 \, [g\, d^{-1}]. \tag{37}$$

We shall obtain the following Tabulation for a density of the particles of $\sigma = 4$:

s_{max} [cm]	$10^{-2.5}$	10^{-2}	$10^{-1.5}$	10^{-1}
M [t/d]	0.37	3.7	37	370

To this, there will have to be added a few t's as mass contribution of the large meteors. The values found are rather somewhat too low in relation to the value that is being considered most likely today (10^3 t/d and more; F.L. Whipple, 1952).

But, when we transform the meteor frequencies as indicated by F.G. Watson (1939,1956) and others (cf.also, e.g., A.C.B. Lovell, 1954, Chapter VII) and as dependent on the luminosity into frequencies as dependent on the radius by means of a mean relation between radius and visual luminosity, then the frequency will lie for 10^{-1}cm - such particles may still be considered meteors - under the value indicated in the Equation (36), by five powers of ten. For that reason, it is not possible to join equation (36) with the frequency of the radii into one general law governing the meteors.

The model of the distribution function of the orbital elements as considered here is - being a first attempt and as such, is still somewhat schematic. The following characteristic joint properties of the orbits of zodiacal light particles and of meteors, however, come clearly to the fore: The majority of the orbits have small inclinations. The main contribution of the density and of the meteors as well is supplied by particles having a large semi-axis of $>$ 1 A.U., in the perihelion. The high frequency of the medium and large eccentricities of the meteors as an apparent effect due to the "geometry" of the catchment of the particles that is also due to the varying velocity of incidence.- The possibility cannot be precluded altogether that meteors and zodiacal light particles are a group of particles within relatively limited intervals of the orbital elements ($a \approx 2$-5 A.U., $e = 0.6$-0.9, $i = 0$-20°).- on account of the findings described in the preceding Section, several investigators (H.C.van de Hulst, 1947; H.K. Kallmann, 1955) postulated two different components of the interplanetary dust: a zodiacal light component on circular orbits, and a meteoric component on orbits of greater eccentricity. On the basis of the comparison of the frequency of the orbital elements as carried out here, this postulate appears to be, at least, over-refined: the distribution of inclination and the distribution of the perihelion distances are probably similar for large and small particles. An absence of large particles with small eccentricity and large semi-axis near 1 A.U. may be an apparent effect, because these particles are to be looked for in the weaker meteors, due to their smaller relative velocity; otherwise, such an absence can be explained easily by the lesser action of the Poynting-Robertson effect.

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